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Optimization algorithms of operative control in water distribution systems

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Abstract

This paper discusses a multilevel algorithm for finding optimal control in a static distribution system based on the idea of aggregation technique. We present mathematical model of this system with its elements as well as two basic algorithms. The first is a simulation algorithm of the pipeline network and the other is an algorithm for finding an optimal control at the pumping station. This paper discusses the static problem (Kotowski and Olesiak, 1980) of energy wastes minimization in the water network and also describes an algorithm for solving it. Finally, an algorithm of operative control of the water distribution systems is presented.

Keywords: Distribution networks; Flows in networks; Simulation; Optimization; Optimal control

AMS classification: 65K10; 65Y20; 76M25

1. Introduction

The main goal of water distribution networks is to fulfill the demand of receivers. To achieve that it is necessary do deliver the appropriate amount of water (with the determined quality and in the specified time intervals). The network consists of pipelines which connect sources (e.g., pump stations) with consumers.

The multilevel control structure of such a system has been discussed in [2–4]. Moreover, the most often used optimization criterion is that of electrical energy cost minimization, because its operation and maintenance may be included in the capital costs. The multilevel control structure for this kind of system has been discussed in [2–4].

The proposed model of a control system assumes [1, 2, 7] that the important aim of control is to satisfy the requirements of water consumers. Water distribution systems are designed to deliver

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water from pump stations to water consumers through pipeline networks equipped with a variety of components-pump stations, valves, reservoirs.

According to [9,10,13] we assume that in our model of water distribution system, cost minimization is the basic optimization criterion. In framework the control system proposed for implementation is based on a three level structure [3, 13].

The first level is that of pump units direct control, regulating valves, heads and flow of a network. On this level, based on a control algorithm, the actual number of working pump units as well as the desired position of regulating values are determined. Whereas the recomended values of head and flow from pumping stations are received (as parameters of control algorithm) from the second (upper) level.

This one determines the above mentioned parameters which ensures the implementation of receivers' demand. The actual value of this parameters we obtain from electricity cost minimization. As a result, we obtain a graphic schedule which illustrate the cooperation between pump stations and reservoirs.

The highest, third level determines an optimal graphic schedule for filling the reservoirs. This is based on the forecasted water histogram of the consumers' demands as well as the parameters of the aggregated water distribution system. The algorithm ensures the fulfilment of the consumer demands and the minimization of the energy costs taking into consideration the varying prices for electrical energy.

The results of our investigations in this study are related to the second level of the control system. The aims of this level is to maintain the water distribution system in an optimal regime of action presuming that the desired demands of the consumers will be fulfilled.

The mathematical model of optimal control in the water distribution network should take into account the following assumptions:

- random demands of consumers,
- dynamic structure caused by the parameters of reservoir,
- structure and topography of the network,
- nonlinear characteristics of pumps and pipelines,
- time-varying prices of electrical energy.

The optimization problem includes the values of flows in pipeline, the heads in the network nodes, the positions of valves and pumps to be operated in every moment of considered period of time. This yields to the dynamic and nonlinear problem of high dimension, which is practically unsolvable. Therefore, in our previous papers we had to use some special techniques based on the aggregation and decomposition methods, connected with some necessary simplifications of the mathematical model [13], to achieve finally the three-level optimization algorithm.

2. Model description

A mathematical model of water distribution system [5, 7, 8, 12] consists of the following elements:

- model of the pump stations,
- model of the pipeline,
- model of the pipelines network,
- model of the reservoir.

Now we describe the above elements of the mathematical model in details. Let IP be a number of the pumping stations. We have assumed that at each i station $\bar{z}_{i_p}$ pumps with identical characteristics are set up. When one of these pumps works it consumes electricity and

$$P(y) = \alpha + \beta y, \quad \alpha, \beta \geqslant 0 \tag{1}$$

is the general formula that connects the consumed electricity with the output flow y. For pump stations one obtains the following formulas:

$$P_{i_p}(y_{i_p}, z_{i_p}) = \alpha_{i_p} z_{i_p} + \beta_{i_p} y_{i_p}; \quad i_p = 1, \dots, \mathrm{IP} \tag{2}$$

which is true, when z_{i_p} pumps work at it, thus the summarized output flow is y_{i_p}. Let H_{i_p} denotes the work characteristics:

$$H_{i_p}(y_{i_p}, z_{i_p}) = H^0_{i_p} - G_{i_p}\left(\frac{y_{i_p}}{z_{i_p}} - y^0_{i_p}\right)^2; \quad i_p = 1, \dots, \mathrm{IP} \tag{3}$$

which are the relations between the number of working pumps z_{i_p}, the summarized output flow y_{i_p} and the head of water H_{i_p}. In formula (3) we have $H^0_{i_p}$, $G_{i_p} > 0$. It is clear that $0 \leqslant z_{i_p} \leqslant \bar{z}_{i_p}$. A total output flow from the station is restricted by the technical conditions in the way presented below:

$$z_{i_p} \underline{y}_{i_p} \leqslant y_{i_p} \leqslant z_{i_p} \bar{y}_{i_p}; \quad i_p = 1, \dots, \mathrm{IP} \tag{4}$$

where $\underline{y}_{i_p}$ and $\bar{y}_{i_p}$ denote, respectively, a minimal and maximal feasible output flow from a single pump. Let $v_{i_p}(y_{i_p})$ denotes the desired head at the output of the pump station. An output flow y can be realized when the following constraints are met:

$$H_{i_p}(y_{i_p}, z_{i_p}) \geqslant v_{i_p}(y_{i_p}); \quad i_p = 1, \dots, \mathrm{IP}. \tag{5}$$

When (5) is true the differences of head $H_{i_p}(y_{i_p}, z_{i_p}) - v_{i_p}(y_{i_p})$ can be compensated by the appropriate valves value v_{i_p} whenever necessary.

In our model each arc (a pipeline) i_a can be described by the flow y_{i_a}, $i_a = 1, 2, \dots, \mathrm{IA}$. The head difference x_{i_a} between two ends of the pipeline is given by the following formula:

$$x_{i_a} = k_{i_a} y^2_{i_a} \operatorname{sgn}(y_{i_a}) + d_{i_a}; \quad i_a = 1, \dots, \mathrm{IA} \tag{6}$$

where d_{i_a} is a difference of elevations and k_{i_a} is the resistance of the pipeline. The full description of physical rules in the pipelines network is known owing to the Kirchhoff's laws:

- The first law — material continuity at a node

$$Ay = \sigma \tag{7}$$

where $y \in \mathscr{R}^{\mathrm{IA}}$, σ is a vector of consumers' demand, A is an incidence matrix [IC × IA] order and IC is a number of consumer nodes. We assume that consumers' demand are realized in network's nodes only.

- The second law — the loop equations

$$Bx = 0 \tag{8}$$

where B is a loop matrix and $x \in \mathscr{R}^{\mathrm{IA}}$ is a vector of the head differences x_{i_a}.

3. Simulation algorithm of water distribution network

One of the most important problems in the operative control of the water distribution systems [6, 12] is the determination of flows and heads in the water supply network. Our algorithm is interactive and therefore it is essential to solve the problem (6)–(8) many times for varying values of σ_{i_c}, $i_c = 1,\ldots,\mathrm{IC}$. There are several well-known methods for solving the flow and the head in distribution networks [1, 6–8, 12]. We have developed a method based on the theory of network flow. For the given values σ_{i_c}, $i_c = 1,\ldots,\mathrm{IC}$ Eqs. (6)–(8) may be transformed into a problem of energy wastes minimization in the water supply network. The algorithm elaborated by us and presented here is designed to solve the following task of a static optimization with linear constraints:

$$f(y) = \sum_{i_a=1}^{\mathrm{IA}} f_{i_a}(y_{i_a}) \tag{9}$$

subject to

$$Ay = \sigma. \tag{10}$$

The objective function in (9) has its interpretation as the power wastes in the analyzed network for the water distribution. Its detailed components are defined as follows:

$$f_{i_a}(y_{i_a}) = k_{i_a} y_{i_a}^3 \operatorname{sgn}(y_{i_a}) + d_{i_a} y_{i_a}; \quad i_a = 1,\ldots,\mathrm{IA}. \tag{11}$$

Formula (11) results from Bernoulli's law. One may see that $f_{i_a}(y_{i_a}) = x_{i_a} y_{i_a}$ or using vector notation, $f(y) = x^{\mathrm{T}} y$. The algorithm designated to solve the task (9)–(10) consists of three basic parts. The introductory part is designed to transform the initial problem into the problem without constraints. The second part deals with the problem of determining the searching direction and in the third we cope with the problem of minimizing at the given direction.

3.1. Transformation of the water distribution network model

The transformation of the problem (9)–(10) into the problem without constraints is based on disentanglements of the constraints (10). Formally, the final results may be presented as follows:

$$y = B^{\mathrm{T}} y_l + D\sigma. \tag{12}$$

The B matrix consists of the elements 0, 1, -1. One may prove that its detailed form is always given as follows:

$$B = [I, B_l] \tag{13}$$

where I is a unity matrix. So, (13) is a definition of B_l and (12) is a definition of D. The algebra of matrices shows that such a D and B_l must exist. In our program a corresponding segment executes the renumeration of the variables so that the components of the vector y_l become the first components of the vector y. It determines the $D\sigma$ vector's components and informs the compact form about the position of the nonzero elements of the matrix B. Thus, the determination of all the components of vector y, on the basis of (12), requires only the adding and subtracting operation to be carried out. Taking into account (12) in the objective function (9), the initial problem (9)–(10) should be replaced by the task of static optimization without constraints in the form of

$$f(B^{\mathrm{T}} y_l + D\sigma) \rightarrow \min. \tag{14}$$

The dimension of the problem (14), i.e., the number of components of the vector y_l is the difference between the number of variables and the numbers of constraints in the initial problem and will be denoted by IR. According to (9)–(10) the objective function in the reduced problem is a strictly convex and a twice-differentiable function. It may be easily shown that the b gradient and H matrix of Hesse of the objective function in (14) are expressed as follows:

$$b = B \cdot \nabla f(y) \tag{15}$$

$$H = B \cdot \Lambda \cdot B^{\mathrm{T}} \tag{16}$$

where Λ is diagonal matrix in the form

$$\Lambda = \mathrm{diag}\{6k_{i_a} |y_{i_a}|;\ i_a = 1, \ldots, \mathrm{IA}\}. \tag{17}$$

The following parts of the algorithm are the procedures for determining the searched direction and for minimization of the objective function at the given direction. A detailed description of these procedures is presented in the subsequent sections.

3.2. Algorithm for determining a new direction of searching based on the modified Newton's method

The problem of determination of a new feasible direction for search in the modified Newton's method is simplified to solve the linear-equation system

$$H \cdot q = -b \tag{18}$$

where H is the Hessian of reduced objective function and b the gradient determined at the point y is the actual optimal solution. The peculiarity of Eq. (18) induced the authors to try such a method for its solution, which could enable the elimination of the operations of multiplication by zero in the computational program. Finally, the problem (18) has been replaced by the equivalent task of optimization in the form:

$$F(q) = \frac{1}{2} \cdot q^{\mathrm{T}} H \cdot q + b^{\mathrm{T}} \cdot q \rightarrow \min \tag{19}$$

and this problem has been solved using the Fletcher–Reeves's method. The effectivity of such an approach depends on particular properties of task (19). The general outline of the Fletcher–Reeves's algorithm is as shown below:

Algorithm 1

Step 1. Let $i = 1$, $q_0 = 0$, $q_i = -b$.

Step 2. Determine the minimum of the objective function in the direction $q_0 + \tau q_i$:

$$F(q_0 + \phi_i q_i) = \min_{\tau > 0} F(q_0 + \tau q_i) \tag{20}$$

Now let $q_0 = q_0 + \phi_i q_i$.

Step 3. If $i > \mathrm{IR}$ then Stop. Otherwise determine the gradient of the function F at the point q_0:

$$g_i = \nabla F(q_0) = Hq_0 + b. \tag{21}$$

Step 4. Determine a new direction q_{i+1} according to the equation:

$$q_i = -g_i + \frac{\langle g_i, g_i \rangle}{\langle g_{i-1}, g_{i-1} \rangle} \cdot q_{i-1} \tag{22}$$

put $i = i + 1$ and go to Step 2.

The solution of the problem (20) according to (22) is brought to the solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} F(q_0 + \tau \cdot q_i) = 0. \tag{23}$$

Hence, we obtain

$$q_0^{\mathrm{T}} \cdot H \cdot q_i + \tau \cdot q_i^{\mathrm{T}} \cdot H \cdot q_i + b^{\mathrm{T}} \cdot q_i = 0. \tag{24}$$

Because q_0 and q_i are conjugate directions, it is obtained from the definition:

$$\tau_i = -b^{\mathrm{T}} q_i / q_i^{\mathrm{T}} H q_i. \tag{25}$$

The determination of a new feasible direction for searching q_0 for our initial task (19) requires, in case of application of the Fletcher–Reeves's algorithm, to bear certain computational expenditures. The operations connected with the determinations of products (25) as well as Hq_0 in (21) are critical. Application of formula (16) results in the first case:

$$q_i^{\mathrm{T}} H q_i = (B^{\mathrm{T}} q_i)^{\mathrm{T}} \Lambda \cdot (B^{\mathrm{T}} q_i) = \sum_{j=1}^{\mathrm{IA}} \lambda_j v_{ij}^2; \quad i = 1, \ldots, \mathrm{IR} \tag{26}$$

where $v_i = B^{\mathrm{T}} q_i$; $i = 1, \ldots, \mathrm{IR}$. Because B is a matrix with elements 0, 1, -1 the determination of successive component of the vector v_i requires only the operation of addition and subtraction. In order to calculate the product $q^{\mathrm{T}} H q$ according to (26) only $2 \cdot \mathrm{IA}$ multiplications are required, where IA is the number of arcs of the analyzed network. A similar analysis may be carried out for Eq. (22):

$$H \cdot q_0 = B \cdot \Lambda \cdot B^{\mathrm{T}} \cdot q_0 = B \cdot w_0 \tag{27}$$

where $w_0 = \Lambda B^{\mathrm{T}} v_0$; $v_0 = B^{\mathrm{T}} q_0$. Because Λ is a diagonal matrix for the determination of product Hq_0 it requires, according to (27), only IA multiplications.

3.3. Analysis of the procedure properties for searching the minimum in the given direction

The successive searching stage for current minimum of the objective function $f(y)$ is based on the minimization process of the one variable function $G(t)$ defined by the equation

$$G(t) = f(y(t)) \tag{28}$$

where

$$y(t) = y_0 + t \cdot q_0 \tag{29}$$

is the parametric equation of a straight line in R^{IA}. According to the differentiation principles of a function with many variables one obtains

$$G'(t) = \sum_{i_a=1}^{IA} d_{i_a} q_{0i_a} + 3 \sum_{i_a=1}^{IA} k_{i_a} (y_{0i_a} + t \cdot q_{0i_a})^2 \operatorname{sgn}(y_{0i_a} + t \cdot q_{0i_a}) \cdot q_{0i_a}. \tag{30}$$

In an similar way as in the case of formula (30) one may determine the second derivative of the function $G(t)$:

$$G''(t) = 6 \sum_{i_a=1}^{IA} k_{i_a} |y_{0i_a} + t_a \cdot q_{0i_a}| \cdot q_{0i_a}^2. \tag{31}$$

As a result of formula (31) the chart of function $G''(t)$ is a pointwise linear function. More precisely, $G(t)$ is a stochastic process: $G(t) = G(t, w)$, where w is an element from the random space $\Omega : w \in \Omega$. Space Ω respond for the random form of the consumers' demand. One may show that condition $G''(t) > 0$ is satisfied for any t with a probability equal to 1 (i.e., $P(A_G) = 1$, where $A_G = \{w \in \Omega : G''(t, w) > 0\}$ and P is a probability measure on Ω). Its derivative $G'(t)$ is a strictly increasing function consisting of parabolic segments. From (30) the following conclusion may be drawn:

$$\lim_{t \to \infty} G'(t) = +\infty. \tag{32}$$

Hence and from (22) it results that $G'(t)$ has exactly one root t_0 which is the searched minimum of the function $G(t)$. Additionally, $t_0 > 0$. The above described properties enable the elaboration of an efficient algorithm for the determination of the position of minimum towards t.

The idea of the algorithm is based on the determination of two sequential ruptures of function $G'(t)$ between which is the minimum as well as the values of the function $G'(t)$ at these points t' and t''. Then one must determine the value of the function $G'(t)$ at any point, e.g., $t = (t' + t'')/2$. Through these three points one should carry out a parabola and determine its roots. The sketched algorithms enables to define with computer accuracy the position of the point t_0. The procedure for determination of points t' and t'' is based on the idea of the set of points discontinuity of the function $G'''(t)$. According to Eq. (30), t' is the point of discontinuity of the function $G'''(t)$, if

$$(\exists i_a = 1, \ldots, IA)\ (q_{0i_a} \neq 0,\ y_{i_a} + t' q_{0i_a} = 0). \tag{33}$$

Such a method can be implemented by constructing the first set N_1 defined by

$$N_1 = \{\exists i_a = 1, \ldots, IA;\ q_{0i_a} \neq 0,\ y_{i_a}/q_{0i_a} < 0\} \tag{34}$$

because, the extreme $G(t)$ should be searched for $t > 0$. Having determined t_0, the new current solution is found:

$$\bar{y}_{i_a} = y_{i_a} + t_0 \cdot q_{0i_a}, \quad i_a = 1, \ldots, IA. \tag{35}$$

This finishes the process of determination of the successive minimum of the objective function $f(y)$.

3.4. Numerical analysis

The complexity of the Fletcher–Reeves algorithm based on the principles presented above, measured by the number of multiplication is of order of IR · IA. The classical approach relying on the solution of system (18) is characterized by the complexity of order of $(\mathrm{IR})^3$. The version which enables the optimization of the network consisting of about 100 arcs has no more than 19 kbytes of memory. For our model consisting of IN = 76 nodes and IA = 96 arcs (also one reservoir and three pump stations) we require no more than a few seconds for determining the optimal flow with the IBM compatible PC.

4. Operative control in water distribution systems

Aggregated network models have been described by many authors [8, 13]. In the model of the water distribution system the following properties have been taken into account:

- The proper working system depends upon the realization of certain relations between heads at network nodes. The possibility of filling or emptying reservoirs is particularly important.
- The pump stations are controlled discretely by turning on and off the pumps one after another. The pumps have to operate with a maximum efficiency.
- The cost of electricity consumed by pump stations in a given period is a goal function.

The optimization problems thus obtained are so complicated that, in practice, the attempts to solve larger systems are practically impossible [1,8,12]. The difficulties can be overcome by using problem decomposition and the aggregation of the most complex elements of the system (e.g., the pipeline network). The approach presented here, differs from the other formulations [2,8] because it takes into account the heads relations in the network, furthermore, the discrete character of pump stations yields to a more precise description of the system, but this creates additional computational problems.

The optimization algorithm has two levels. At the upper level a dynamic problem is solved. Its result is a schedule of the reservoirs' exploitation. The data about flows to the reservoirs are submitted to the lower level, where a static problem is solved. Its solution determines how many pumps should be turned on in the pump stations and their current yields. The subject of this part is the description of the algorithm for solving the static problem. It exploits branch-and-bound method [11, 13] as well.

4.1. Formulation of the static problem

The system under consideration includes IP pumping stations, IS reservoirs and IC consumers. Vector r has the following form: $r=(-y,q,\sigma)$. Optimization problem of the lower level consists of minimization of the energy costs used up by pumping stations:

$$F(q,\sigma)=\min_{z,y}(\alpha\cdot z+\beta\cdot y) \tag{36}$$

subject to

$$H_{i_\mathrm{p}}(y_{i_\mathrm{p}},z_{i_\mathrm{p}})\geqslant v_{i_\mathrm{p}}(r), \qquad i_\mathrm{p}=1,\dots,\mathrm{IP}, \tag{37}$$

$$h_{i_s} - k_{i_s} \cdot q_{i_s}^2 \geqslant v_{i_s}(r) \text{ if } q_{i_s} < 0, \quad i_s = 1, \ldots, \mathrm{IS} \tag{38}$$

$$h_{i_s} + k_{i_s} \cdot q_{i_s}^2 \geqslant v_{i_s}(r) \text{ if } q_{i_s} \geqslant 0, \quad i_s = 1, \ldots, \mathrm{IS} \tag{39}$$

$$\underline{y}_{i_p} \cdot z_{i_p} \leqslant y_{i_p} \leqslant \bar{y}_{i_p} \cdot z_{i_p}, \quad i_p = 1, \ldots, \mathrm{IP} \tag{40}$$

$$0 \leqslant z_{i_p} \leqslant \bar{z}_{i_p} \text{ and integer for } \quad i_p = 1, \ldots, \mathrm{IP} \tag{41}$$

$$\beta = (\beta_1, \beta_2, \ldots, \beta_I)^{\mathrm{T}}$$

$$\sum_{i_p=1}^{\mathrm{IP}} y_{i_p} = \sum_{i_s=1}^{\mathrm{IS}} q_{i_s} + \sum_{i_c=1}^{\mathrm{IC}} \sigma_{i_c} = y_0. \tag{42}$$

For a given flow vector r, the value of pressure head $v(r)$ can be determined by solving a set of nonlinear network Eqs. (6)–(8). However, this procedure is time consuming (the system may contain hundreds and thousands of equations and variables) and besides it requires the determination of the arcs parameters of the network. The aggregation of the network model introduced by [13] consists of approximations of the relations $v(r)$ by means of quadratic forms:

$$v_{i_p}(r) = r^{\mathrm{T}} \cdot \tilde{C}^{i_p} \cdot r + \tilde{D}^{i_p} \cdot r + \tilde{E}^{i_p}; \quad i_p = 1, \ldots, \mathrm{IP}. \tag{43}$$

For any $i_p = 1, \ldots, \mathrm{IP}$ one may write $v_{i_p}(r) = v_{i_p}(r, \tilde{C}^{i_p}, \tilde{D}^{i_p}, \tilde{E}^{i_p})$ to express that the values of these functions depends on the element of the previously unknown matrices $\tilde{C}^{i_p}, \tilde{D}^{i_p}, \tilde{E}^{i_p}$. All of their elements may be determined by means of the least-squares-approximation method. An algorithm for obtaining the aggregated model and for estimating its accuracy has been described in [13]. Parameters $\tilde{C}^{i_p}, \tilde{D}^{i_p}, \tilde{E}^{i_p}$ of the aggregated model can be determined on the basis of a passive identification experiment in particular. In this case, heads and flows must be measured at the pump stations and reservoir. The use of relation (43) in (37), involves the following optimization problem:

$$F(q, \sigma) = \min_{z, y}(\alpha \cdot z + \beta \cdot y) \tag{44}$$

subject to

$$y^{\mathrm{T}} \cdot C^{i_k}(z) \cdot y + D^{i_k}(z) \cdot y + E^{i_k} \geqslant 0; \quad i = 1, \ldots, \mathrm{IK}. \tag{45}$$

The parameters $C^{i_k}, D^{i_k}, E^{i_k}$ results from taking into account formula (3) and assumption that component vectors q, σ of vector r are fixed.

$$\underline{y}_{i_p} \cdot z_{i_p} \leqslant y_{i_p} \leqslant \bar{y}_{i_p} \cdot z_{i_p}; \quad i_p = 1, \ldots, \mathrm{IP} \tag{46}$$

$$0 \leqslant z_{i_p} \leqslant \bar{z}_{i_p} \quad \text{and integer for } i_p = 1, \ldots, \mathrm{IP}, \tag{47}$$

$$\sum_{i_p=1}^{\mathrm{IP}} y_{i_p} = y_0. \tag{48}$$

In (43) $\mathrm{IK} = \mathrm{IP} + 2\,\mathrm{IS}$. Formula (45) contains all constraints (37)–(39) from the previous problem (36)–(42). One may see that according to the initial assumptions $H_{i_p}(y_{i_p}, z_{i_p})$ are quadratic forms of their arguments. The problem formulated above is that of a nonlinear mixed-integer programming one. A method of solving it is discussed below.

4.2. A method for solving the optimization problem of the water distribution system

The problem formulated on the basis of the aggregated model has a number of variables and constraints that are proportional to the number of pump stations and reservoirs in the system. Usually this is smaller than the number of variables and constraints that can occur when the original system is investigated.

The algorithm is based on the conception of the branch-and-bound method [11]. The constraints cause a set of solutions that are incoherent and may contain no more than $\prod_{i_p=1}^{IP}(1+\bar{z}_{i_p})$ subareas. It cannot be assumed that the quadratic forms (43) are seminegative. The functions cannot be concave.

This property of problem (44)–(48) makes its solution more difficult. As it is well known there are no general and effective methods of solving the problems of the type described above. This involved the authors to develop an algorithm specially designated for solving the problems of this class [5, 6, 13]. In analyzing problem (44)–(48) one may observe that when the vector of integer variables z is fixed to an acceptable value, the obtained problem is dependent only on continuous variables y. The problem will be denoted as $P(z)$:

$$P(z)=\sum_{i_p=1}^{IP}\alpha_{i_p}\cdot z_{i_p}+\min_{y}\left\{\sum_{i_p=1}^{IP}\beta_{i_p}\cdot y_{i_p}\right\} \tag{49}$$

subject to

$$y^{\mathrm{T}}\cdot C^{i_k}(z)\cdot y+D^{i_k}(z)\cdot y+E^{i_k}(z)\geqslant 0;\quad i=1,\dots,\mathrm{IK}, \tag{50}$$

$$\underline{y}_{i_p}\cdot z_{i_p}\leqslant y_{i_p}\leqslant\bar{y}_{i_p}\cdot z_{i_p};\quad i_p=1,\dots,\mathrm{IP}, \tag{51}$$

$$\sum_{i_p=1}^{IP}y_{i_p}=y_0. \tag{52}$$

Summing up both sides of constraints (51), one obtains

$$\sum_{i_p=1}^{IP}\underline{y}_{i_p}z_{i_p}\leqslant\sum_{i_p=1}^{IP}y_{i_p}\leqslant\sum_{i_p=1}^{IP}\bar{y}_{i_p}z_{i_p}. \tag{53}$$

And taking into consideration relation (52) results in the following inequalities:

$$\sum_{j=1}^{IP}\underline{y}_j z_j\leqslant y_0\quad\text{and}\quad\sum_{j=1}^{IP}\bar{y}_j z_j\geqslant y_0. \tag{54}$$

Checking whether formula (54) is true for a certain value z is called a feasibility test for the problem $P(z)$. When any of the inequalities (54) is not satisfied, the problem $P(z)$ cannot have feasible solutions, and thus it cannot determine the optimal solution.

The relaxation of constraints (50)–(52) of the problem $P(z)$ is used when the problem of estimating the value of the objective function (49) occurs.

Let $Y(z)$ be a set of feasible solutions of the problem $P(z)$. $Y(z)$ is defined by constraints (50)–(52). Let $Y_L(z)$, in turn, be a set obtained from $Y(z)$ by leaving out the nonlinear constraints (50).

$$Y_L(z) = \left\{ y \in \mathscr{R}^{\mathrm{IP}} \mid \underline{y}_j z_j \leqslant y_j \leqslant \bar{y}_j z_j; \quad j = 1,\dots,\mathrm{IP}; \ \sum_{j=1}^{\mathrm{IP}} y_j = y_0 \right\}. \tag{55}$$

Obviously, $Y(z) \subset Y_L(z)$, hence

$$f(z) = \alpha z + \min_{y \in Y_L(z)} \beta y \leqslant \alpha z + \min_{y \in Y(z)} \beta y = F(z). \tag{56}$$

The function $f(z)$ defined above can be used in order to estimate lower bound of the value of $F(z)$, since $f(z) \leqslant F(z)$ for each z. The estimated value $f(z)$ can be determined by solving the linear programming problem:

$$f(z) = \alpha z + \min_{y \in Y_L(z)} \beta z. \tag{57}$$

The above problem can be easily transformed into a knapsack problem with constrained variables. As it is shown in [11] a problem like this has an analytical solution, which considerably simplifies the estimating procedure.

Let E be the set of all feasible discrete vectors $z=(z_1,\dots,z_l)$ (according to (47)). A list of ordered couples $\langle z^j, f(z^j)\rangle$ (where $z^j \in E$) representing subproblems $P(z^j)$ is obtained as a first step of the algorithm. The list includes only the subproblems which have passed the feasibility test and thus for which $Y_L(z^j) \neq \emptyset$:

$$L = \{\langle z^j, f(z^j)\rangle \mid Y_L(z^j) \neq \emptyset;\ z^j \in E\}. \tag{58}$$

At the second step, from the list L the subproblem of the last estimate value $f(z^k)$ is selected:

$$f(z^k) = \min_j \{ f(z^j) \mid \langle z^j,\ f(z^j)\rangle \in L\}. \tag{59}$$

The subproblem $P(z^k)$ is deleted from the list L and nonlinear programming problem (49)–(52) corresponding to it is solved. When there are no feasible solutions ($Y(z^k) \neq \emptyset$) or when the obtained value of the $F(z^k)$ is greater than $F(\tilde{z})$, where $\tilde{z}$ is currently the best feasible solution (at the beginning $F(\tilde{z} = \infty)$, one must pass onto the selection of another subproblem from the list. When the obtained solution z^k is better than the so far best $\tilde{z}$ ($F(z^k) < F(\tilde{z})$), then $\tilde{z}$ must be replaced by z^k ($\tilde{z} = z^k$), and then all of the subproblems $P(z^j)$ which estimates $f(z^j)$ are greater than $F(\tilde{z})$ (so for which $f(z^j) \geqslant F(\tilde{z})$), must be deleted from the list L.

$$L = L - \{\langle z^j, f(z^j)\rangle \mid f(z^j) \geqslant F(\tilde{z}),\ \langle z^j,\ f(z^j)\rangle \in L\}. \tag{60}$$

This procedure allows the reduction of a number of problems in the list L and it is called the optimality test. The problems for which $f(z^j) \geqslant F(\tilde{z})$ can be deleted from the list because even without solving them we can state that their solutions cannot be better than $\tilde{z}$ (because of (56)).

The procedure is repeated until the list L contains any elements.

When $L = \emptyset$, the algorithm ends up and $\tilde{z}$ is the optimal solution (it may turn out that the problem does not have feasible solution).

Using the feasibility and optimality tests we may state that certain $P(z)$ problems can not give optimal solution without the necessity of solving them. The branch-and-bound method presented here consists of two stages. In the first stage a list of all problems $P(z)$, which may potentially give an optimal solution, is created. The problems $P(z)$ that are assigned to the list, are selected on the basis of the feasibility tests (55). In the second stage we find the solution of the problem with the best estimation value. Next we delete from the list these problems for which, there is no doubt that, their solution cannot be better.

5. Conclusions

It must be noted that the problem presented above is quite different from the ones known as inventory problems [10]. The classical model of this problem concerns the isolated reservoir or a system of them. In water distribution systems such an isolation would be regarded as an oversimplification

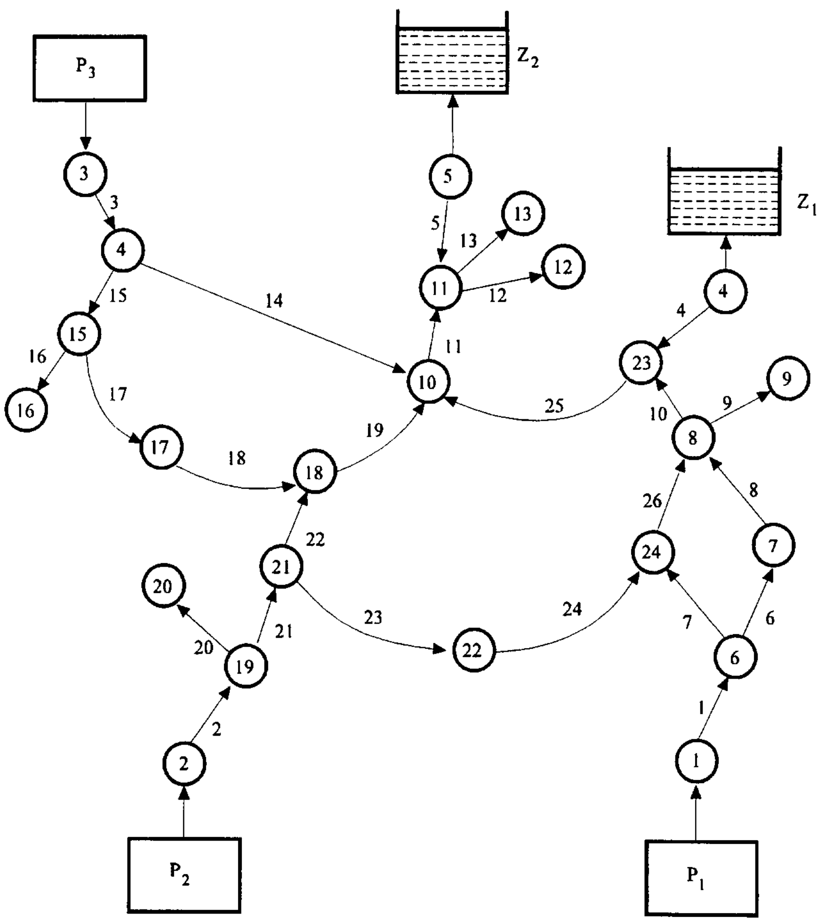

Fig. 1. Exemplary water distribution system.

of the problem because there are strong connections between water levels in a reservoir and flows and heads in the network. This should not be neglected, otherwise it leads to the achievement of some physically unrealized variants.

The results of the tests confirm the usefulness of the presented aggregated network model. This algorithm can be used as a lower stage of the dynamic problem solving procedure for the determination of an optimal schedule $q(t)$ for filling the reservoir. We used the dynamic programming method for the systems with one reservoir. For systems with more than one reservoir the algorithm based on the Wolfe method is more appropriate.

Appendix

In order to verify the numerical effectiveness of the developed algorithms of operative control, series of 40 tests have been solved. In addition, some simulation tests for water distribution network equipped with three pump stations and two reservoirs have been also carried out.

The elaborated tests of feasibility and optimality enabled the rejection of nearly 90% of the problems $P(z)$ without the necessity of solving them. The obtained nonlinear programming problem $P(z)$ is solved by means of a modified cutting planes Kelley's method developed by the authors [9]. The aim of the modification was to take into consideration the possibility of nonconvexity of the feasible solutions set.

Exemplary system (Fig. 1) consist of three pump stations (P_1, P_2, P_3), two reservoirs (Z_1, Z_2), 24 nodes and 26 arcs.

The parameters of the network are given from Table 1 whereas the pump stations and reservoirs parameters from Table 2.

Table 1
Network parameters

Nodes IN = 24			Pipelines IA = 26		
i	h_i	q_i	i	k_i	d_i
1	0.0	−579	1	0.100E−06	4.0
2	4.0	−578	2	0.140E−05	5.0
3	13.0	−578	3	0.250E−05	0.0
4	18.0	0	4	0.600E−05	−2.0
5	21.0	0	5	0.100E−04	−2.0
6	4.0	174	6	0.900E−04	11.0
7	15.0	147	7	0.100E−03	7.0
8	15.0	16	8	0.500E−04	0.0
9	15.0	180	9	0.190E−03	0.0
10	14.0	45	10	0.140E−05	1.0
11	14.0	50	11	0.500E−05	0.0
12	17.0	154	12	0.100E−03	3.0
13	15.0	96	13	0.500E−04	1.0
14	13.0	74	14	0.100E−03	1.0
15	13.0	87	15	0.160E−04	5.0

Table 1 (*Contd.*)

16	15.0	38	16	0.790E−03	2.0
17	7.0	128	17	0.100E−02	−6.0
18	7.0	57	18	0.370E−04	0.0
19	9.0	37	19	0.220E−03	7.0
20	10.0	74	20	0.180E−03	1.0
21	10.0	96	21	0.100E−05	1.0
22	1.0	164	22	0.370E−04	−3.0
23	16.0	38	23	0.650E−04	−9.0
24	11.0	80	24	0.540E−03	10.0
			25	0.410E−03	−2.0
			26	0.110E−03	4.0

Note: h_i — the altitude of ith node,
q_i — the desired flow in ith node,
k_i — the resistance of ith pipeline,
d_i — the difference of ith pipeline ends altitude.

Table 2
Pump station and reservoir parameters

Pump stations IP = 3								Reservoirs IS = 2			
i	$\bar{z}_i$	$\underline{y}_i$	$\bar{y}_i$	H_i^0	k_i	α_i	β_i	i	$\bar{s}_i$	h_i	k_i
1	2	192	292	67.4	3×10^{-4}	159 991	197	1	7500	30	10^{-6}
2	4	178	444	72.9	1.6×10^{-4}	137 220	380	2	7500	30	10^{-6}
3	2	132	192	59.8	5.6×10^{-4}	79 920	107				

Note: The symbols are the same as in the Section 2.
$\bar{s}_i$ — the capacity of ith reservoir,
h_i — the height of ith reservoir in relation to connection node,
k_i — the resistance of ith reservoir connection,
For the other symbol interpretation see formulas (2), (3) and (6).

The results of the simulation algorithm are given from Table 3.

The results of the above-mentioned tests (Section 4) are shown in Table 4 (where reservoirs carry a function of sources) and Table 5 (where reservoirs carry a function of receivers).

The names of the columns (Tables 4 and 5) correspond to the notions in Section 2. So we notice
σ_0 — the consumer's demand,
q_i — the reservoirs flows,
y_0 — the total output flow from the pump stations,
z^* — the vector of optimal pumps configuration,

Table 3
Results of network simulation

Nodes IN = 24		Pipelines IA = 26		
i	v_i	i	y_i	x_i
1	55.9	1	577.0	4.0
2	51.6	2	573.0	5.5
3	48.6	3	579.0	0.8
4	33.1	4	0.0	−2.0
5	35.4	5	0.0	−2.0
6	51.9	6	222.7	15.5
7	36.4	7	180.4	10.3
8	36.1	8	75.8	0.3
9	30.0	9	179.9	6.1
10	37.9	10	−3.4	1.0
11	37.4	11	299.8	0.4
12	32.0	12	153.9	5.4
13	35.9	13	95.9	1.5
14	47.8	14	298.6	9.9
15	47.1	15	206.4	0.7
16	43.9	16	38.0	3.1
17	46.5	17	81.5	0.6
18	46.5	18	−46.5	−0.1
19	46.1	19	87.5	8.7
20	44.1	20	74.0	2.0
21	44.9	21	467.1	1.2
22	51.8	22	191.0	−1.7
23	35.1	23	180.2	−6.9
24	41.6	24	16.3	10.1
		25	−41.4	−2.7
		26	116.7	5.5

Note: y_i — the flow in ith pipeline,
x_i — the pressure head difference in ith pipeline,
v_i — the potential of the ith node.

y_1^*, y_2^*, y_3^* — the optimal output flow from each pump stations,
F_{opt} — the minimal energy, which enables the realization of consumer's demand,
DL — the number of problems L list, after feasibility test,
LRZ — the number of noneliminated problems from L list,
T_0 — the computing time.

Here accuracy of 10^{-5} is assumed. Analyzing the times T_0 of solving the test, one can say that our algorithms can be on-line implemented to the static optimization of water distribution system.

Table 4
Results of 20 tests for optimization algorithms of operative control (reservoirs carry a function of sources)

Nr	σ_0 (1/s)	q_1 (1/s)	q_2 (1/s)	y_0 (1/s)	z^*	y_1^* y_2^* y_3^*	F_{opt} (kW)	DL	LRZ	T_0 (s)
1	500	100	−100	300	(002)	0 0 300	191	2	1	9
2	750	−100	−100	550	(011)	0 362 188	374	6	1	10
3	1000	−100	−100	800	(111)	292 316 192	575	9	2	12
4	1500	−250	−250	1000	(112)	292 327 381	679	11	1	3
5	1500	−300	−300	900	(111)	292 416 192	613	10	1	12
6	750	−150	−150	450	(101)	259 0 192	311	3	1	14
7	1000	−150	−150	700	(012)	0 374 326	473	8	1	15
8	1800	−300	−300	1200	(121)	290 722 188	865	13	3	42
10	900	−100	−200	600	(011)	0 424 176	396	6	1	17
11	2000	−300	−300	1400	(221)	488 734 178	1068	16	7	41
12	2000	−400	−400	1200	—	—	—	13	13	45
13	1000	−300	−300	400	(010)	0 400 0	289	3	1	5
14	300	−50	0	250	(100)	250 0 0	209	1	1	3
15	300	−100	0	200	(100)	200 0 0	199	1	1	3
16	300	−150	0	150	(001)	0 0 150	95	1	1	3

Table 4 (*Contd.*)

17	600	−200	0	400	(010)	0 400 0	289	3	1	4
18	600	−300	0	300	(002)	0 0 300	191	2	1	2
19	600	0	−300	300	(002)	0 0 300	191	2	1	5
20	2500	−400	−400	1700	—	—	—	15	15	46

Table 5
Results of 20 tests for optimization algorithms of operative control (reservoirs carry a function of receivers)

Nr	σ_0 (1/s)	q_1 (1/s)	q_2 (1/s)	y_0 (1/s)	z^*	y_1^* y_2^* y_3^*	F_{opt} (kW)	DL	LRZ	T_0 (s)
1	300	0	0	300	(010)	0 300 0	251	2	2	7
2	200	100	0	300	(010)	0 300 0	251	2	2	7
3	300	50	0	350	(010)	0 350 0	270	3	2	3
4	300	150	0	450	—	—	—	3	3	6
5	300	200	0	500	(102)	227 0 273	393	4	2	14
6	300	100	0	400	(101)	273 0 163	304	3	2	3
7	300	100	100	500	(102)	201 0 299	391	4	2	2
8	300	0	200	500	(011)	0 352 148	366	4	1	4
9	300	250	0	550	(110)	239 311 0	462	6	5	9
10	500	100	100	700	(111)	243 299 157	555	8	5	11

Table 5 (*Contd.*)

11	300	400	0	700	(210)	413 287 0	647	8	8	12
12	500	100	250	850	(112)	240 334 275	660	10	5	11
13	750	100	100	950	(211)	435 360 156	766	11	8	13
14	500	250	250	100	(212)	422 308 270	846	11	10	25
15	1000	100	100	1200	(221)	437 606 157	1007	13	13	23
16	500	100	300	900	(121)	195 550 149	780	10	10	29
17	500	200	200	900	(211)	419 330 152	761	10	10	33
18	300	100	250	650	(111)	219 278 153	542	7	6	17
19	1000	200	200	1400	—	—	—	13	13	37
20	1000	300	300	1600	—	—	—	16	16	38

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